J. Math. Anal. Appl. 364 (2010) 195–203

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces ☆

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A R T I C L E   I N F O

Article history:
Received 18 January 2009
Available online 14 October 2009
Submitted by R. Gornet

Keywords:
Locally symmetric Lorentz space
Constant scalar curvature
Spacelike hypersurface
Second fundamental form

A B S T R A C T

The purpose of this paper is to study compact or complete spacelike hypersurfaces with constant normalized scalar curvature in a locally symmetric Lorentz space satisfying some curvature conditions. We give an optimal estimate of the squared norm of the second fundamental form of such hypersurfaces. Furthermore, the totally umbilical hypersurfaces are characterized.

1. Introduction and main results

Let L_1^{n+1} be an $(n+1)$-dimensional Lorentz space, i.e., a pseudo-Riemannian manifold of index 1. A hypersurface M^n of a Lorentz space is said to be spacelike if the induced metric on M^n from that of the Lorentz space is positive definite. When the Lorentz space is of constant curvature, we call it Lorentz space form. As is well known, for $n \geqslant 2$, the de Sitter space $S_1^{n+1}(1)$ is the standard simply connected Lorentzian space form of positive constant sectional curvature 1.

Spacelike hypersurfaces of $S_1^{n+1}(1)$ with constant mean curvature have been under very extensive study since Goddard [9] posed the conjecture that every complete spacelike hypersurface in S_1^{n+1} with constant mean curvature H must be totally umbilical. It is well known that this conjecture is false in general and holds only for some special cases. See Akutagawa [1], Montiel [12] and Ramanathan [16] for details. For a more complete study of spacelike hypersurfaces in general Lorentzian space with constant mean curvature, we refer to [13] and references therein.

Another natural Goddard-like problem is to study hypersurfaces of Lorentzian space with constant scalar curvature. An interesting result of Cheng and Ishikawa [6] states that the totally umbilical round spheres are the only compact spacelike hypersurfaces in $S_1^{n+1}(1)$ with constant normalized scalar curvature $R < 1$. Some other authors, such as Brasil, Colares and Palmas [3], Camargo, Chaves and Sousa Jr. [4], Caminha [5], Hu, Scherfner and Zhai [10] and Li [11] have also worked on related problems.

It is important and natural to study complete spacelike hypersurfaces with constant scalar curvature in the more general Lorentz spaces since they have important meaning in the relativity theory and are of substantial interest from geometric and mathematical cosmology points of view. First of all, we recall that, for constants c_1 and c_2, Choi et al. [8,17] introduced the class of $(n+1)$-dimensional Lorentz spaces L_1^{n+1} of index 1 which satisfy the following two conditions (here and in the sequel, K denotes the sectional curvature on L_1^{n+1}):

☆ This work was supported by the NNSF of China (No. 10871138).

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0022-247X/$ – see front matter © 2009 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2009.10.029

(i) for any spacelike vector u and any timelike vector v

$$K(u,v) = -\frac{c_1}{n}, \tag{1.1}$$

(ii) for any spacelike vectors u and v

$$K(u,v) \geqslant c_2. \tag{1.2}$$

It is obvious that the Lorentz space form $L_1^{n+1}(c)$ satisfies (1.1) and (1.2) for $-(c_1/n) = c_2 = c$. There are several examples of Lorentz spaces which are not Lorentz space forms and satisfy (1.1) and (1.2). For instance, semi-Riemannian product manifold $H_1^k(-c_1/n) \times N^{n+1-k}(c_2)$, $c_1 > 0$, and $\mathbf{R}_1^k \times S^{n+1-k}(1)$. In particular, $\mathbf{R}_1^1 \times S^n(1)$ is a so-called *Einstein Static Universe*. Also the *Robertson–Walker spacetime* $N(c,f) = I \times_f N^3(c)$ is another general example of Lorentz space, where I denotes an open interval of $\mathbf{R}_1^1$ and $f > 0$ a smooth function defined on the interval I, $N^3(c)$ a 3-dimensional Riemannian manifold of constant curvature c. $N(c,f)$ also satisfies (1.1) and (1.2) if we choose an appropriate function f. For more details, we refer the readers to [8,17].

In this paper, we study compact or complete spacelike hypersurface M^n with constant normalized scalar curvature R in a locally symmetric Lorentz space satisfying curvature conditions (1.1) and (1.2). Hereafter, Lorentz space L_1^{n+1} is called *locally symmetric* if all the covariant derivative components $\bar{R}_{ABCD;E}$ of the Riemannian curvature tensor $\bar{R}$ of L_1^{n+1} vanish.

In order to present our main results, we need some basic facts and notations. First of all, denote by $\bar{R}_{CD}$ the components of the Ricci tensor of L_1^{n+1} satisfying (1.1) and (1.2), then the scalar curvature $\bar{R}$ of L_1^{n+1} is given by

$$\bar{R} = \sum_{A=1}^{n+1} \epsilon_A \bar{R}_{AA} = -2\sum_{i=1}^{n} \bar{R}_{(n+1)ii(n+1)} + \sum_{i,j=1}^{n} \bar{R}_{ijji} = 2c_1 + \sum_{i,j=1}^{n} \bar{R}_{ijji}.$$

On the other hand, it is well known that the scalar curvature of a locally symmetric Lorentz space is constant. Hence, $\sum_{i,j=1}^{n} \bar{R}_{ijji}$ is constant.

Now, for spacelike hypersurface M^n with constant normalized scalar curvature R in a locally symmetric Lorentz space satisfying curvature conditions (1.1) and (1.2), we know that $\sum_{i,j=1}^{n} \bar{R}_{ijji} - n(n-1)R$ is a constant. Noticing that the relation between the squared length S of the second fundamental form and the mean curvature H of M^n in L_1^{n+1}: $n(n-1)R = \sum_{j,k} \bar{R}_{kjjk} - n^2H^2 + S$ (see Eq. (2.3) in Section 2), it follows that $n^2H^2 - S = \sum_{j,k} \bar{R}_{kjjk} - n(n-1)R$ is a constant. This fact suggests us to define a constant P such that

$$n(n-1)P = n^2H^2 - S = \sum_{i,j=1}^{n} \bar{R}_{ijji} - n(n-1)R. \tag{1.3}$$

Using this constant P, we can finally establish our main results: Theorem 1.1 deals with the compact case while Theorem 1.2 solves the complete case.

Theorem 1.1. *Let M^n $(n \geqslant 3)$ be a compact spacelike hypersurface with constant normalized scalar curvature R in an $n+1$-dimensional locally symmetric Lorentz space L_1^{n+1} satisfying* (1.1) *and* (1.2)*. Suppose that the constant P defined by* (1.3) *satisfies $0 \leqslant P \leqslant \frac{2c}{n}$ and $c = 2c_2 + \frac{c_1}{n} > 0$, where c_1, c_2 given as in* (1.1), (1.2)*. Then $\sup S = nH^2$ and M^n is totally umbilical.*

Remark 1. If L_1^{n+1} is the de Sitter space $S_1^{n+1}(c)$, then $-\frac{c_1}{n} = c_2 = c = 1$ and $P = 1 - R$ following from (1.3). At that time, the assumption $0 \leqslant P \leqslant \frac{2c}{n}$ in Theorem 1.1 becomes $\frac{n-2}{n} \leqslant R \leqslant 1$ and Theorem 1.1 reduces to the Li's result [11, Theorem 4.3]. It is not known whether does the condition $0 \leqslant P \leqslant \frac{2c}{n}$ in Theorem 1.1 essential? As we know, this interesting problem has not been settled even in de Sitter spaces.

For the complete case, we will prove the following theorem:

Theorem 1.2. *Let M^n $(n \geqslant 3)$ be a complete spacelike hypersurface with constant normalized scalar curvature R in an $n+1$-dimensional locally symmetric Lorentz space L_1^{n+1} satisfying* (1.1) *and* (1.2)*. Denote $c = 2c_2 + \frac{c_1}{n}$ and c_1, c_2 given as in* (1.1), (1.2)*. Suppose that M^n has bounded mean curvature H:*

(i) *If $c > 0$ and $0 \leqslant P \leqslant \frac{2c}{n}$, where the constant P defined by* (1.3)*, then $\sup S = nH^2$ and M^n is totally umbilical.*
(ii) *If $c > 0$ and $\frac{2c}{n} < P \leqslant c$, then either $\sup S = nP$ and M^n is totally umbilical, or*

$$\sup S \geqslant S_{\min},$$

where $S_{\min} = \frac{n}{(n-2)(nP-2c)}[n(n-1)P^2 - 4c(n-1)P + nc^2]$. Furthermore, if $H^2 < \frac{(n-2)c^2}{n(nP-2c)}$, then $S_{\min} > nH^2$ and M^n is a non-totally umbilical hypersurface.

Remark 2. In particular, let $L_1^{n+1} = S_1^{n+1}(1)$, as we pointed out in Remark 1, the assumption $0 \leqslant P \leqslant \frac{2c}{n}$ becomes $\frac{n-2}{n} \leqslant R \leqslant 1$. So Theorem 1.2(i) can be viewed as a kind of extension of the result due to F.E.C. Camargo et al. in [4], saying that a complete spacelike hypersurface M^n ($n \geqslant 3$) in the de Sitter space $S_1^{n+1}(c)$ with constant normalized scalar curvature R satisfying $\frac{n-2}{n}c \leqslant R \leqslant c$ must be totally umbilical provided that M^n has bounded mean curvature.

On the other hand, consider the spacelike hypersurface embedded into $S_1^{n+1}(1)$ given by $T_{1,r} = \{x \in S_1^{n+1}(1) \mid -x_0^2 + x_1^2 = -\sinh^2 r\}$ with r a positive real number. It follows from [10] that $T_{1,r}$ is non-totally umbilical with normalized scalar curvature satisfying $0 < R = \frac{n-2}{n}(1 - \tanh^2 r) < \frac{n-2}{n}$. Hence, the assumption $0 \leqslant P \leqslant \frac{2c}{n}$ in Theorem 1.2(i) is essential.

Remark 3. When we take $L_1^{n+1} = S_1^{n+1}(1)$, then $P = 1 - R$. A straightforward calculation gives $S_{\min} = \frac{(n-1)(n-2-nR)}{n-2} + \frac{n-2}{n-2-nR}$. Meanwhile, the condition $\frac{2c}{n} < P \leqslant c$ in Theorem 1.2(ii) reduces to $0 \leqslant R < \frac{n-2}{n}$. The squared length of the second fundamental form of $T_{1,r}$ is exactly $S_{\min}$. This means that the estimate of S in Theorem 1.2(ii) is best possible.

2. Preliminaries

In this section, we will calculate $\triangle S$ and $\square(nH)$ for spacelike hypersurfaces in locally symmetric Lorentz spaces satisfying (1.1) and (1.2). Then we prove a key Lemma 2.4 in the proof of Theorems 1.1 and 1.2. We shall make use of the following convention for the indices: $1 \leqslant A, B, C, \ldots \leqslant n+1$, $1 \leqslant i, j, k, \ldots \leqslant n$.

2.1. Setting for general Lorentz space and its spacelike hypersurface

Let M^n be a spacelike hypersurface of Lorentz space L_1^{n+1}. We choose a local field of pseudo-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+1}\}$ in L_1^{n+1} such that, at each point of M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n and e_{n+1} is the unit timelike normal vector. Denote by $\{\omega_A\}$ the corresponding dual coframe and by $\{\omega_{AB}\}$ the connection forms of L_1^{n+1}. Then the structure equations of L_1^{n+1} are given by

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0, \quad \epsilon_i = 1,\ \epsilon_{n+1} = -1,$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \epsilon_C \epsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D.$$

The components $\bar{R}_{CD}$ of the Ricci tensor and the scalar curvature $\bar{R}$ of L_1^{n+1} are given by

$$\bar{R}_{CD} = \sum_B \epsilon_B \bar{R}_{BCDB}, \qquad \bar{R} = \sum_A \epsilon_A \bar{R}_{AA}.$$

The components $\bar{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\bar{R}$ are defined by

$$\sum_E \epsilon_E \bar{R}_{ABCD;E} \omega_E = d\bar{R}_{ABCD} - \sum_E \epsilon_E (\bar{R}_{EBCD}\omega_{EA} + \bar{R}_{AECD}\omega_{EB} + \bar{R}_{ABED}\omega_{EC} + \bar{R}_{ABCE}\omega_{ED}).$$

Restricting these forms to the spacelike hypersurface M^n in L_1^{n+1}, we have $\omega_{n+1} = 0$, so $\sum_i \omega_{(n+1)i} \wedge \omega_i = 0$. By Cartan's lemma, there are h_{ij} such that $\omega_{(n+1)i} = \sum_j h_{ij}\omega_j$ and $h_{ij} = h_{ji}$. This gives the second fundamental form of M^n, $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$, and its squared length, $S = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H is defined by $H = \frac{1}{n}\sum_j h_{jj}$. The connection forms $\{\omega_{ij}\}$ of M^n are characterized by the structure equations of M^n:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

Using the structure equations we obtain Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}). \tag{2.1}$$

Components R_{ij} of Ricci tensor and scalar curvature R of M^n are given by

$$R_{ij} = \sum_k \bar{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.2}$$

$$n(n-1)R = \sum_{j,k} \bar{R}_{kjjk} - n^2H^2 + S. \tag{2.3}$$

Let h_{ijk}, h_{ijkl} denote the first and the second covariant derivatives of h_{ij}, respectively, so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki},$$
$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

Then we obtain the Codazzi equation and the Ricci identity:

$$h_{ijk} - h_{ikj} = \bar{R}_{(n+1)ijk},$$
$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl}.$$

Restricting the covariant derivative $\bar{R}_{ABCD;E}$ of $\bar{R}_{ABCD}$ on M^n, then $\bar{R}_{(n+1)ijk;l}$ is given by

$$\bar{R}_{(n+1)ijk;l} = \bar{R}_{(n+1)ijkl} + \bar{R}_{(n+1)i(n+1)k}h_{jl} + \bar{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \bar{R}_{mijk}h_{ml}, \tag{2.4}$$

where $\bar{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\bar{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\sum_l \bar{R}_{(n+1)ijkl}\omega_l = d\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk}\omega_{li} - \sum_l \bar{R}_{(n+1)ilk}\omega_{lj} - \sum_l \bar{R}_{(n+1)ijl}\omega_{lk}.$$

The Laplacian $\triangle h_{ij}$ of h_{ij} is defined by $\triangle h_{ij} = \sum_k h_{ijkk}$. Using Gauss equation, Codazzi equation, Ricci identity and (2.4), a straightforward calculation will give

$$\begin{aligned}\triangle h_{ij} = {} & (nH)_{ij} + \sum_k (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) - nH\sum_l h_{il}h_{lj} + Sh_{ij} \\ & - \sum_k (h_{kk}\bar{R}_{(n+1)ij(n+1)} + h_{ij}\bar{R}_{(n+1)k(n+1)k}) \\ & - \sum_{k,l}(2h_{kl}\bar{R}_{lijk} + h_{jl}\bar{R}_{lkik} + h_{il}\bar{R}_{lkjk})\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\frac{1}{2}\triangle S = {} & \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\triangle h_{ij} \\ = {} & \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}(\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j})h_{ij} \\ & - \left(\sum_{i,j} nHh_{ij}\bar{R}_{(n+1)ij(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\right) \\ & - 2\sum_{i,j,k,l}(h_{kl}h_{ij}\bar{R}_{lijk} + h_{il}h_{ij}\bar{R}_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij} + S^2.\end{aligned} \tag{2.6}$$

2.2. *Estimate of $\triangle S$ and $\square(nH)$*

All the equations presented in Subsection 2.1 hold on hypersurfaces in a general Lorentz space L_1^{n+1}. From now on, we assume further that L_1^{n+1} is a locally symmetric Lorentz space satisfying curvature conditions (1.1) and (1.2). In such a special class of ambient spaces, we will estimate the right-hand side of (2.6) by using the local symmetry and the curvature conditions (1.1) and (1.2). In order to do this, we will choose $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$, then local symmetry of L_1^{n+1} implies that

$$\sum_{i,j,k}(\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j})h_{ij} = 0. \tag{2.7}$$

Using curvature conditions (1.1) and (1.2), we get

$$-\left(nH\sum_{i,j}h_{ij}\bar{R}_{(n+1)ij(n+1)}+S\sum_{k}\bar{R}_{(n+1)k(n+1)k}\right)=-\left(nH\sum_{k}\lambda_k\bar{R}_{(n+1)kk(n+1)}-S\sum_{k}\bar{R}_{(n+1)kk(n+1)}\right)$$
$$=\sum_k(S-nH\lambda_k)\frac{c_1}{n}=c_1\big(S-nH^2\big). \tag{2.8}$$

Notice that $S-nH^2=\frac{1}{2n}\sum_{j,k}(\lambda_j-\lambda_k)^2$ (see Eq. (2.15)), we also have

$$-2\sum_{i,j,k,l}(h_{kl}h_{ij}\bar{R}_{lijk}+h_{il}h_{ij}\bar{R}_{lkjk})=-2\sum_{j,k}\big(\lambda_j\lambda_k-\lambda_k^2\big)\bar{R}_{kjjk}\geqslant c_2\sum_{j,k}(\lambda_j-\lambda_k)^2$$
$$=2c_2\big(nS-n^2H^2\big). \tag{2.9}$$

Substituting (2.7), (2.8) and (2.9) into (2.6), we finally obtain the following lemma.

Lemma 2.1. *Let M^n be a spacelike hypersurface in locally symmetric Lorentz space satisfying curvature conditions* (1.1) *and* (1.2)*, then*

$$\frac{1}{2}\triangle S\geqslant\sum_{i,j,k}h_{ijk}^2+\sum_i\lambda_i(nH)_{ii}+(2nc_2+c_1)\big(S-nH^2\big)+\left(S^2-nH\sum_i\lambda_i^3\right), \tag{2.10}$$

where λ_i, $1\leqslant i\leqslant n$, are principal curvatures of M^n.

According to Cheng and Yau [7], we introduce the self-adjoint operator $\Box$ acting on any C^2-function f by $\Box f=\sum_{i,j}(nH\delta_{ij}-h_{ij})f_{ij}$. Taking $f=nH$ on M^n, it follows from (2.3) that

$$\Box(nH)=\sum_{i,j}(nH\delta_{ij}-h_{ij})(nH)_{ij}$$
$$=\frac{1}{2}\triangle(nH)^2-\sum_i(nH)_i^2-\sum_i\lambda_i(nH)_{ii}$$
$$=\frac{1}{2}\triangle\left(\sum_{i,j}\bar{R}_{ijji}-n(n-1)R\right)+\frac{1}{2}\triangle S-n^2|\nabla H|^2-\sum_i\lambda_i(nH)_{ii}. \tag{2.11}$$

By (1.3), we know that $\sum_{i,j}\bar{R}_{ijji}-n(n-1)R=\text{const.}$ for hypersurface M^n in L_1^{n+1} with constant normalized scalar curvature R. Then, substituting (2.10) into (2.11), we have

$$\Box(nH)\geqslant\sum_{i,j,k}h_{ijk}^2-n^2|\nabla H|^2+(2nc_2+c_1)\big(S-nH^2\big)+\left(S^2-nH\sum_i\lambda_i^3\right). \tag{2.12}$$

We will need the following algebraic Lemma 2.2 obtained by M. Okumura [14] and whose proof can be found in [2], and an asymptotic maximum principle (Lemma 2.3) at infinity for complete manifolds due to H. Omori [15].

Lemma 2.2. *Let $\mu_1,\dots,\mu_n$ be real numbers such that $\sum_i\mu_i=0$ and $\sum_i\mu_i^2=\beta^2$, where $\beta\geqslant 0$ is constant. Then*

$$\left|\sum_i\mu_i^3\right|\leqslant\frac{n-2}{\sqrt{n(n-1)}}\beta^3,$$

and equality holds if and only if at least $n-1$ of the μ_i's are equal.

Lemma 2.3. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $F:M^n\to\mathbb{R}$ be a smooth function which is bounded from above on M^n. Then there exists a sequence of points $\{x_k\}\in M^n$ such that*

$$\lim_{k\to\infty}F(x_k)=\sup F,$$
$$\lim_{k\to\infty}\big|\nabla F(x_k)\big|=0,$$
$$\lim_{k\to\infty}\sup\max\big\{\big(\nabla^2F(x_k)\big)(X,X)\colon |X|=1\big\}\leqslant 0.$$

2.3. Key lemma

The following lemma will play a crucial role in the proofs of Theorems 1.1 and 1.2.

Lemma 2.4. *Let L_1^{n+1} be a locally symmetric Lorentz space satisfying* (1.1) *and* (1.2) *and M^n $(n \geqslant 3)$ be a spacelike hypersurface with constant normalized scalar curvature R.*

(i) *If $P \geqslant 0$, then*

$$|\phi|^2 L_H(\phi) \leqslant \Box(nH), \tag{2.13}$$

where $|\phi|^2 = S - nH^2$ and $L_H(\phi) = nc - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|$, $c = 2c_2 + \frac{c_1}{n} > 0$.

(ii) *If $P \geqslant 0$ and the mean curvature H of M^n is bounded, then there is a sequence of points $\{x_k\} \in M^n$ such that*

$$\begin{aligned} &\lim_{k\to\infty} nH(x_k) = \sup(nH),\\ &\lim_{k\to\infty} \left|\nabla(nH)(x_k)\right| = 0,\\ &\lim_{k\to\infty} \sup\bigl(\Box(nH)(x_k)\bigr) \leqslant 0. \end{aligned} \tag{2.14}$$

Proof. (i) Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. Let $\mu_i = \lambda_i - H$ and denote $|\phi|^2 = \sum_i \mu_i^2$. A direct computation gives

$$|\phi|^2 = S - nH^2 = \frac{1}{2n}\sum_{i,j}(\lambda_i - \lambda_j)^2. \tag{2.15}$$

Consequently, $|\phi|^2 = 0$ if and only if M^n is totally umbilical. We also have

$$\sum_i \lambda_i^3 = nH^3 + 3H\sum_i \mu_i^2 + \sum_i \mu_i^3.$$

By applying Lemma 2.2 to real numbers $\mu_1, \dots, \mu_n$, we get

$$\begin{aligned} -nH\sum_i \lambda_i^3 &= -n^2H^4 - 3nH^2\sum_i \mu_i^2 - nH\sum_i \mu_i^3\\ &\geqslant 2n^2H^4 - 3nSH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\bigl(S - nH^2\bigr)^{\frac{3}{2}}. \end{aligned}$$

Putting the above inequality into (2.12), we obtain

$$\Box(nH) \geqslant \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + |\phi|^2 L_H(\phi). \tag{2.16}$$

In the following, we will prove that $\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 \geqslant 0$. Indeed, as P is a constant, differentiating formula (1.3) exteriorly yields $n^2 H H_k = \sum_{i,j} h_{ij} h_{ijk}$, then Cauchy–Schwarz inequality leads to

$$\sum_k n^4 H^2 (H_k)^2 = \sum_k \left(\sum_{i,j} h_{ij} h_{ijk}\right)^2 \leqslant \left(\sum_{i,j} h_{ij}^2\right)\left(\sum_{i,j,k} h_{ijk}^2\right),$$

namely, $n^4 H^2 |\nabla H|^2 \leqslant S\sum_{i,j,k} h_{ijk}^2$. Together with the fact $n^2H^2 - S \geqslant 0$ since $P \geqslant 0$, we conclude that $\sum_{i,j,k} h_{ijk}^2 \geqslant n^2|\nabla H|^2$. Therefore, (2.13) follows from (2.16).

(ii) Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. By definition, $\Box(nH) = \sum_i (nH - \lambda_i)(nH)_{ii}$. As $P \geqslant 0$, it follows from (1.3) that $\lambda_i^2 \leqslant S \leqslant n^2H^2$, which shows that

$$0 \leqslant n|H| - |\lambda_i|. \tag{2.17}$$

Using (1.2), (2.1) and (2.17), we have $R_{ijij} \geqslant c_2 - \lambda_i\lambda_j \geqslant c_2 - n^2H^2$. This shows that the sectional curvatures of M^n are bounded from below because H is bounded. Therefore we may apply Lemma 2.3 to the function nH, and obtain a sequence of points $\{x_k\} \in M^n$ such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH), \qquad \lim_{k\to\infty}\left|\nabla(nH)(x_k)\right| = 0, \qquad \lim_{k\to\infty}\sup\bigl(nH_{ii}(x_k)\bigr) \leqslant 0. \tag{2.18}$$

Since H is bounded, taking subsequences if necessary, we can arrive to a sequence $\{x_k\} \in M^n$ which satisfies (2.18) and such that $H(x_k) \geqslant 0$ (by changing the orientation of M^n if necessary). Thus from (2.17) we get

$$0 \leqslant nH(x_k) - \left|\lambda_i(x_k)\right| \leqslant nH(x_k) - \lambda_i(x_k) \leqslant nH(x_k) + \left|\lambda_i(x_k)\right| \leqslant 2nH(x_k). \tag{2.19}$$

Using once more the fact that H is bounded, from (2.19) we infer that $\{nH(x_k) - \lambda_i(x_k)\}$ is non-negative and bounded. By applying $\Box(nH)$ at x_k, taking the limit and using (2.18) and (2.19) we have

$$\lim_{k\to\infty} \sup\bigl(\Box(nH)(x_k)\bigr) \leqslant \sum_i \lim_{k\to\infty} \sup\bigl((nH - \lambda_i)(x_k) nH_{ii}(x_k)\bigr) \leqslant 0.$$

This completes the proof of Lemma 2.4. $\Box$

3. Proofs of theorems

Proof of Theorem 1.2. According to Lemma 2.4(ii), there exists a sequence of points $\{x_k\} \in M^n$ such that

$$\lim_{k\to\infty} \sup\bigl(\Box(nH)(x_k)\bigr) \leqslant 0, \qquad \lim_{k\to\infty} nH(x_k) = \sup(nH). \tag{3.1}$$

From (1.3) and (2.15), we have

$$|\phi|^2 = n(n-1)\bigl(H^2 - P\bigr). \tag{3.2}$$

Notice that $\lim_{k\to\infty}(nH)(x_k) = \sup(nH)$ and P is a constant, (3.2) implies that

$$\lim_{k\to\infty} |\phi|^2(x_k) = \sup|\phi|^2. \tag{3.3}$$

Evaluating (2.13) at the points x_k of the sequence, taking the limit and using (3.1), we obtain that

$$\begin{aligned} 0 &\geqslant \lim_{k\to\infty} \sup\bigl(\Box(nH)(x_k)\bigr) \\ &\geqslant \sup|\phi|^2\left(nc - n\sup H^2 + \sup|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\sup|H|\sup|\phi|\right). \end{aligned} \tag{3.4}$$

Consider the following polynomial given by

$$L_{\sup H}(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\sup|H| x + n\bigl(c - \sup H^2\bigr). \tag{3.5}$$

We shall prove that $L_{\sup H}(\sup|\phi|) > 0$.

If $\sup H^2 < \frac{4(n-1)}{n^2}c$, it is easy to check that the discriminant of $L_{\sup H}(x)$ is negative. Hence, for any x, $L_{\sup H}(x) > 0$, so does $L_{\sup H}(\sup|\phi|) > 0$.

Suppose that $\sup H^2 \geqslant \frac{4(n-1)}{n^2}c$ and let α be the biggest root of the equation $L_{\sup H}(x) = 0$, it is obvious that $\alpha > 0$. Noticing the assumption $P \leqslant \frac{2c}{n}$, then (3.2) leads to

$$\bigl(\sup|\phi|\bigr)^2 = \sup|\phi|^2 \geqslant (n-1)\bigl(n\sup H^2 - 2c\bigr). \tag{3.6}$$

By virtue of (3.6), we can easily check that

$$\sup|\phi|^2 - \alpha^2 \geqslant \frac{n-2}{2(n-1)}\Bigl(n^2\sup H^2 - n\sup H\sqrt{n^2\sup H^2 - 4(n-1)c} - 2(n-1)c\Bigr). \tag{3.7}$$

Therefore, $\sup|\phi|^2 - \alpha^2 > 0$ if and only if

$$n^2\sup H^2 - n\sup H\sqrt{n^2\sup H^2 - 4(n-1)c} - 2(n-1)c > 0. \tag{3.8}$$

Taking into account that the inequality (3.8) is equivalent to $4(n-1)^2c^2 > 0$, which is actually true because of $c > 0$. So

$$\sup|\phi|^2 - \alpha^2 > 0,$$

i.e., $\sup|\phi| > \alpha > 0$, which proves that $L_{\sup H}(\sup|\phi|) > 0$.

Hence, we conclude from (3.4) that $\sup|\phi|^2 = 0$, that is $|\phi| = 0$. Thus, we infer that $S = nH^2$ and M^n is totally umbilical. This finishes the proof of the first part of assertions in Theorem 1.2.

Next, we shall prove the second part of the assertions in Theorem 1.2. Let's recall from Lemma 2.4 that

$$\Box(nH) \geqslant |\phi|^2\left(nc - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|\right). \tag{3.9}$$

Using once more (1.3) and (2.15), we rewrite (3.9) as follows

$$\square(nH) \geqslant \frac{n-1}{n}(S-nP)L_P(S), \tag{3.10}$$

where

$$L_P(S) = nc - 2(n-1)P + \frac{n-2}{n}S - \frac{n-2}{n}\sqrt{\big(n(n-1)P+S\big)(S-nP)}.$$

Evaluating (3.10) at the points x_k of the sequence, taking the limit and using (3.1), (3.3), we obtain that

$$\lim_{k\to\infty} S(x_k) = \sup S. \tag{3.11}$$

Notice that (3.1) also holds for the case of $c>0$ and $\frac{2c}{n} < P \leqslant c$, it follows from (3.1) and (3.11) that

$$\begin{aligned} 0 &\geqslant \lim_{k\to\infty} \sup\big(\square(nH)(x_k)\big) \\ &\geqslant \frac{n-1}{n}(\sup S - nP)L_P(\sup S). \end{aligned} \tag{3.12}$$

Combining (1.3) and (2.15), we have $|\phi|^2 = \frac{n-1}{n}(S-nP) \geqslant 0$, i.e., $\sup S \geqslant nP$. Thus, we have from (3.12) that either $\sup S = nP$, that is, $S \equiv nH^2$ and M^n is totally umbilical, or

$$\sup S > nP \quad \text{and} \quad L_P(\sup S) \leqslant 0. \tag{3.13}$$

The latter inequality is nothing but

$$nc - 2(n-1)P + \frac{n-2}{n}\sup S \leqslant \frac{n-2}{n}\sqrt{\big(n(n-1)P+\sup S\big)(\sup S-nP)}. \tag{3.14}$$

Since $P \leqslant c$ and $\sup S > nP$, so $nc - 2(n-1)P + \frac{n-2}{n}\sup S > 0$, which together with (3.14) yields

$$\left[nc - 2(n-1)P + \frac{n-2}{n}\sup S\right]^2 \leqslant \left[\frac{n-2}{n}\sqrt{\big(n(n-1)P+\sup S\big)(\sup S-nP)}\right]^2. \tag{3.15}$$

Simplifying (3.15), it gives

$$(n-2)(nP-2c)\sup S \geqslant n\big[n(n-1)P^2 - 4c(n-1)P + nc^2\big]. \tag{3.16}$$

By assumption $P > \frac{2c}{n}$ and $n \geqslant 3$, it follows from (3.16) that

$$\sup S \geqslant \frac{n}{(n-2)(nP-2c)}\big[n(n-1)P^2 - 4c(n-1)P + nc^2\big],$$

which is the assertion $\sup S \geqslant S_{\min}$ in the second part of Theorem 1.2. Here we point out that $S_{\min} > 0$ since the discriminant of $n(n-1)P^2 - 4c(n-1)P + nc^2$ is negative for $c > 0$ and $n \geqslant 3$.

Finally, if $H^2 < \frac{(n-2)c^2}{n(nP-2c)}$, by the simple algebraic calculation, we can verify that $S_{\min} > nH^2$, which asserts that the spacelike hypersurface M^n is not totally umbilical. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.1. When M^n $(n \geqslant 3)$ is a compact spacelike hypersurface with constant normalized scalar curvature R in an $n+1$-dimensional locally symmetric Lorentz space L_1^{n+1} satisfying Eqs. (1.1) and (1.2), by taking the similar processing as in the proof of Theorem 1.1 on the inequality $L_{\sup H}(\sup|\phi|) > 0$, we can arrive to

$$nc - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi| > 0, \tag{3.17}$$

which jointly with Lemma 2.4 leads to

$$\square(nH) \geqslant |\phi|^2\left(nc - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|\right) \geqslant 0. \tag{3.18}$$

Since M^n is compact and $\square$ is a self-adjoint operator,

$$\int_{M^n} \square(nH)\, dv_M = 0. \tag{3.19}$$

Integrating now (3.18) on M^n, (3.19) implies that

$$0 \geqslant \int_{M^n} |\phi|^2\left(nc - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|\right)dv_M \geqslant 0. \tag{3.20}$$

Hence, we have $|\phi|^2(nc - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|) = 0$, which together with (3.17) yields $|\phi|^2 = 0$. That is, $S = nH^2$ and M^n is totally umbilical. So the proof of Theorem 1.1 is finished. □

Acknowledgments

The authors would like to thank the referee for his kind advice to improve this article.

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